A Super-Integrable Discretization of the Calogero Model

Hideaki Ujino*
Centre de Recherches Mathématiques, Université de Montréal,
Montreal, Quebec, Canada, H3C 3J7

Luc Vinet
Department of Mathematics and Statistics and Department of Physics,
McGill University, Montreal, Quebec, Canada, H3A 2J5
luc.vinet@mcgill.ca

Haruo Yoshida
National Astronomical Observatory of Japan,
Mitaka, Tokyo 181–8588, Japan
h.yoshida@nao.ac.jp

abstract: A time-discretization that preserves the super-integrability of the Calogero model is obtained by application of the integrable time-discretization of the harmonic oscillator to the projection method for the Calogero model with continuous time. In particular, the difference equations of motion, which provide an explicit scheme for time-integration, are explicitly presented for the two-body case. Numerical results exhibit that the scheme conserves all the$(=3)$ conserved quantities of the (two-body) Calogero model with a precision of the machine epsilon times the number of iterations.

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I Introduction

Numerical analysis of dynamical systems has great importance and a wide variety of applications in science and engineering. The elaboration of schemes for numerical analysis has a long and continuous history of studies as well as a rich accumulation of techniques. Whenever one applies numerical analysis to the equations of motion, one must discretize the time-evolution of the dynamical system that is originally described by differential equations because of the lack of the notion of infinity in numerical analysis. This leads to difference equations, which do not usually describe the same dynamical system as the original one: time-discretization is typically accompanied by modification of the original system, which may cause a significant difference in the behavior of the solution from that of the original system, particularly after integration over a long period. Understanding and controlling such modifications are thus important in the quest for more accurate long-time integration in numerical analysis.

The symplectic integration method (see, for instance, refs. [1, 2, 3]), or the symplectic integrator, is one of the time-discretizations that was invented in such a quest. Given a Hamiltonian

*Present address: Gunma National College of Technology, Maebashi, Gunma 371–8530, Japan, ujino@nat.gunma-ct.ac.jp

$H^{(0)}$, it is designed so that its one-step($=\tau$) time evolution gives the exact one-step time evolution of a modified Hamiltonian $\tilde{H} := H^{(0)} + \tau H^{(1)} + \tau^2 H^{(2)} + \cdots$. Since the modified Hamiltonian is conserved by the flow of the symplectic integrator, the fluctuation of the value of the original Hamiltonian $H^{(0)}$, i.e., the total energy of the system with continuous time, is bounded, which is far more favorable than unbounded increase or decrease of the total energy that one usually observes in other non-symplectic discretizations. This is the reason why the symplectic integration method shows better accuracy even after long-time integration.

However, even the symplectic integration method does not usually have modified constants of motion for all the constants of motion of a system, which often causes secular increase or decrease of the values of the constants of motion after long-time integration. For example, non-existence of the modified constants of motion is proved for the two super-integrable models discretized by the symplectic integrator: the two dimensional Harmonic oscillator with integer frequency ratio (except for the isotropic case) and the two-dimensional Kepler problem [4, 5]. The orbits generated by the symplectic integrator are not closed, though those of the exact analytic solutions are closed, indeed.

Thus the following question naturally arises: are there any discretization schemes that preserve the (super-)integrability of (super-)integrable models? Actually, an extensive collection of the known integrable discretizations of integrable models is now available in the monograph [6] that was published in recent years. However, the super-integrable discretization of super integrable models has not been studied that far yet. Quite recently, an affirmative answer to the question on the super-integrable discretization is shown for the Kepler problem for two and three dimensional cases [7, 8], where the integrable discretization of the harmonic oscillator [9] plays an essential role. The above super-integrable discretization conserves all the constants of motion, i.e., the Hamiltonian, the angular momentum and the Runge–Lenz vector, and generates a sequence of discrete points on the orbit of the exact analytic solution of the Kepler problem. And, of course, the orbit with the eccentricity less than unity is closed.

The purpose of this paper is to present a super-integrable discretization of the Calogero model [10, 11],

$$H := \frac{1}{2}\sum_{i=1}^{N}(p_i^2 + \omega^2 x_i^2) + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{a^2}{(x_i - x_j)^2}. \tag{1.1}$$

The real-valued quantities p_i, x_i, ω and a in the Hamiltonian are the canonical momentum and coordinate of the i-th particle, the strength of the external harmonic confinement and the interaction parameter, respectively. The Calogero model ($\omega \neq 0$) with N degrees of freedom (corresponding to the N-body case) is maximally super-integrable in the sense that it has $2N-1$ constants of motion which are independent of each other [12]. The super-integrable structure of the Calogero model is built up by the Lax formulation [13, 14], with which the eigenvalue problem of an oscillating Hermitian matrix [15] is intrinsically involved. We shall present a discretization that preserves the above super-integrable structure of the Calogero model for the general N-body case. It gives, in particular, the explicit form of the difference equations of motion of the Calogero model for the

two-body case that conserves all the three constants of motion as

$$
\begin{aligned}
\Delta_+ x_{1,n} =& \frac{1}{1+\frac{\omega^2\Delta t^2}{4}}\left[p_{1,n} - \frac{\omega^2}{2}x_{1,n}\Delta t\right] + \frac{1}{1+\frac{\omega^2\Delta t^2}{4}}\frac{2a}{x_{1,n}-x_{2,n}}M_{\mathrm{i},n}\Delta t \\
&- 2\Big[\frac{1-\frac{\omega^2\Delta t^2}{4}}{1+\frac{\omega^2\Delta t^2}{4}}x_{1,n} + \frac{1}{1+\frac{\omega^2\Delta t^2}{4}}p_{1,n}\Delta t\Big]M_{\mathrm{r},n}\Delta t \\
&+ \Big[\frac{1-\frac{\omega^2\Delta t^2}{4}}{1+\frac{\omega^2\Delta t^2}{4}}(x_{1,n}+x_{2,n}) + \frac{1}{1+\frac{\omega^2\Delta t^2}{4}}(p_{1,n}+p_{2,n})\Delta t\Big]\big(M_{\mathrm{i},n}^2 + M_{\mathrm{r},n}^2\Delta t^2\big)\Delta t, \\
\Delta_+ p_{1,n} =& -\frac{\omega^2}{1+\frac{\omega^2\Delta t^2}{4}}\left[x_{1,n} + \frac{1}{2}p_{1,n}\Delta t\right] + \frac{1-\frac{\omega^2\Delta t^2}{4}}{1+\frac{\omega^2\Delta t^2}{4}}\frac{2a}{x_{1,n}-x_{2,n}}M_{\mathrm{i},n} \\
&- 2\Big[\frac{1-\frac{\omega^2\Delta t^2}{4}}{1+\frac{\omega^2\Delta t^2}{4}}p_{1,n} - \frac{\omega^2}{1+\frac{\omega^2\Delta t^2}{4}}x_{1,n}\Delta t\Big]M_{\mathrm{r},n}\Delta t \\
&+ \Big[\frac{1-\frac{\omega^2\Delta t^2}{4}}{1+\frac{\omega^2\Delta t^2}{4}}(p_{1,n}+p_{2,n}) - \frac{\omega^2}{1+\frac{\omega^2\Delta t^2}{4}}(x_{1,n}+x_{2,n})\Delta t\Big]\big(M_{\mathrm{i},n}^2 + M_{\mathrm{r},n}^2\Delta t^2\big)\Delta t, \\
\Delta_+ x_{2,n} =& \Delta_+ x_{1,n}\Big|_{1\leftrightarrow 2}, \quad \Delta_+ p_{2,n} = \Delta_+ p_{1,n}\Big|_{1\leftrightarrow 2},
\end{aligned}
\tag{1.2}
$$

with

$$
\begin{aligned}
M_{\mathrm{i},n} :=& \frac{a}{\sqrt{4a^2\Delta t^2 + Y_n^2}}, \quad M_{\mathrm{r},n} := \frac{2a^2}{4a^2\Delta t^2 + Y_n^2 + Y_n\sqrt{4a^2\Delta t^2 + Y_n^2}}, \\
Y_n :=& \Big[1 - \frac{\omega^2\Delta t^2}{4}\Big](x_{1,n}-x_{2,n})^2 + (p_{1,n}-p_{2,n})(x_{1,n}-x_{2,n})\Delta t,
\end{aligned}
$$

where $x_{i,n}$, $p_{i,n}$, $i = 1, 2$, are the coordinate and the momentum of the i-th particle at the n-th discrete time. The symbol Δ_+ denotes the advanced time-difference defined by

$$
\Delta_+ A_n := \frac{A_{n+1} - A_n}{\Delta t}, \tag{1.3}
$$

for an arbitrary variable A_n (e.g., $A_n = x_{i,n}$, $p_{i,n}$) with the discrete time n.

The paper is organized as follows. In section II, we present a brief summary of the projection method [15] (for review, see refs. [16, 17, 18], for example), which gives a solution to the initial value problem of the Calogero model with continuous time. The Lax equations for the Calogero model provide a map from the Calogero model into the matrix-valued harmonic oscillator and also play an essential role in our discretization. Applying the integrable discretization for the harmonic oscillator [9], we discretize the projection method in section III. A discrete analogue of the Lax equations, which we call the dLax equations, is derived as a natural consequence of the discretization. In section IV, we present the explicit forms of the dLax equations of the Calogero model for the two-body case, which are equivalent to the difference equations of motion (1.2). They provide an explicit scheme for the time-integration of the model. Numerical results obtained by our integrable discretization as well as by two other discretization schemes, namely, the symplectic Euler and energy conservation methods, are also presented. Section V is dedicated to the summary and concluding remarks.

II Projection Method

In terms of the Lax pair for the Calogero–Moser model [14],

$$
\begin{aligned}
&L_{ij}(t) := p_i(t)\delta_{ij} + \frac{\mathrm{i}a}{x_i(t) - x_j(t)}(1-\delta_{ij}),\\
&M_{ij}(t) := \sum_{k(\neq i)} \frac{\mathrm{i}a}{\big(x_i(t) - x_k(t)\big)^2}\delta_{ij} - \frac{\mathrm{i}a}{\big(x_i(t) - x_j(t)\big)^2}(1-\delta_{ij}), \qquad i,j = 1,2,\ldots,N,
\end{aligned}
\tag{2.1}
$$

as well as a diagonal matrix

$$
D(t) := \mathrm{diag}\big(x_1(t), x_2(t), \ldots, x_N(t)\big),
$$

the canonical equation of motion for the Calogero model (1.1) can be cast into the Lax form,

$$
\begin{aligned}
&\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M] - \omega^2 D,\\
&\frac{\mathrm{d}D}{\mathrm{d}t} = [D, M] + L,
\end{aligned}
$$

where $[A, B] := AB - BA$, or equivalently,

$$
\frac{\mathrm{d}L^\pm}{\mathrm{d}t} = [L^\pm, M] \pm \mathrm{i}\omega L^\pm,
\tag{2.2}
$$

where

$$
L^\pm(t) := L(t) \pm \mathrm{i}\omega D(t).
\tag{2.3}
$$

The Lax equations (2.2) allow the following relation between the products of L^+ and L^- and the Hamiltonian

$$
\frac{\mathrm{d}}{\mathrm{d}t}\big((L^+)^{l_1}(L^-)^{m_1}\cdots\big) = [(L^+)^{l_1}(L^-)^{m_1}\cdots, M] + \mathrm{i}\omega\big(l_1 + \cdots - (m_1 + \cdots)\big)(L^+)^{l_1}(L^-)^{m_1}\cdots,
$$

for any nonnegative integers $l_1, m_1, \ldots$. Iterated use of the above formula yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\prod_i \mathrm{Tr}\big(\overrightarrow{\prod_j}(L^+)^{l_{i,j}}(L^-)^{m_{i,j}}\big)\Big) &= \sum_i \prod_{i\neq k} \mathrm{Tr}\big(\overrightarrow{\prod_j}(L^+)^{l_{i,j}}(L^-)^{m_{i,j}}\big)\,\mathrm{Tr}\Big[\overrightarrow{\prod_l}(L^+)^{l_{k,l}}(L^-)^{m_{k,l}}, M\Big]\\
&\quad + \sum_{k,l}(l_{k,l} - m_{k,l}) \prod_i \mathrm{Tr}\big(\overrightarrow{\prod_j}(L^+)^{l_{i,j}}(L^-)^{m_{i,j}}\big)\\
&= \sum_{k,l}(l_{k,l} - m_{k,l}) \prod_i \mathrm{Tr}\big(\overrightarrow{\prod_j}(L^+)^{l_{i,j}}(L^-)^{m_{i,j}}\big),
\end{aligned}
$$

where

$$
\overrightarrow{\prod_j} A_j := A_1 A_2 \cdots .
$$

Thus the constants of motion of the Calogero model can be constructed by taking the trace of any products of L^+ and L^- of the following form,

$$\prod_i \mathrm{Tr}\Bigl(\overrightarrow{\prod_j}(L^+)^{l_{i,j}}(L^-)^{m_{i,j}}\Bigr), \quad \text{if} \sum_{i,j} l_{i,j} = \sum_{i,j} m_{i,j}, \tag{2.4}$$

which includes the constants of motion given in ref. [12]. Considering the case $a = 0$ where the matrices $L^\pm$ become diagonal, one can confirm the quantities given eq. (2.4) above include at least (and also at most) $2N-1$ constants of motion that are independent of each other. For example, one confirms

$$\begin{aligned} &C_1 := \mathrm{Tr}L^+\mathrm{Tr}L^- = \Bigl(\sum_{i=1}^N p_i\Bigr)^2 + \omega^2\Bigl(\sum_{i=1}^N x_i\Bigr)^2, \\ &I_1 := \mathrm{Tr}L^+L^- = 2H, \quad I_2 = \mathrm{Tr}(L^+)^2\mathrm{Tr}(L^-)^2, \end{aligned} \tag{2.5}$$

are constants of motion of the Calogero model (1.1) that are independent of each other. Note that the relations satisfied by the Lax pairs [19, 20]

$$\begin{aligned} &[L^+, L^-] = -2\mathrm{i}\omega[L, D] = -2\omega a(T - E), \quad T_{ij} := 1, \ E_{ij} := \delta_{ij}, \\ &M^{\mathrm{t}}[1, \ldots, 1] = 0, \ [1, \ldots, 1]M = 0, \quad \bigl[\Leftrightarrow TM = MT = 0\bigr] \end{aligned} \tag{2.6}$$

particularly the relations in the second line of eq. (2.6) which we call the sum-to-zero property, have played a crucial role in the quantum analogue of the Lax formulation.

The initial value problem of the Calogero model can be solved by the projection method [15], in which the Lax formulation presented above plays a crucial role. This method can be formulated in an analogous way to the Dirac picture in the time-dependent perturbation theory of quantum mechanics. Let us introduce $L^\pm_{\mathrm{D}}$, the $L^\pm$ matrix in the "Dirac picture" by

$$L^\pm(t) =: \mathrm{e}^{\pm\mathrm{i}\omega t}L^\pm_{\mathrm{D}}(t), \quad L^\pm(0) = L^\pm_{\mathrm{D}}(0). \tag{2.7}$$

Then the Lax equation (2.2) is rewritten as

$$\frac{\mathrm{d}L^\pm_{\mathrm{D}}}{\mathrm{d}t} = [L^\pm_{\mathrm{D}}, M],$$

which has the same form as the Heisenberg equation in the Dirac picture with the time-dependent perturbation $M(t) := M\bigl(x_1(t), x_2(t), \ldots, x_N(t)\bigr)$.

The above equation allows the formal solution as

$$L^\pm_{\mathrm{D}}(t) = U^\dagger(0,t)L^\pm(0)U(0,t), \tag{2.8}$$

where the time-evolution unitary matrix $U(t', t)$ is given by the Dyson series of $M(t)$,

$$U(t', t) := \sum_{k=0}^\infty \int_{t'}^t \mathrm{d}t_k \int_{t'}^{t_k} \mathrm{d}t_{k-1} \cdots \int_{t'}^{t_2} \mathrm{d}t_1 M(t_1)\cdots M(t_{k-1})M(t_k), \tag{2.9}$$

which has the semigroup property:

$$
\begin{aligned}
U(0,t) &:= U[\boldsymbol{x}(0),\boldsymbol{p}(0);a,\omega;t] \\
&\quad = U[\boldsymbol{x}(0),\boldsymbol{p}(0);a,\omega;t']U[\boldsymbol{x}(t'),\boldsymbol{p}(t');a,\omega;t-t'] = U(0,t')U(t',t), \\
U^\dagger(0,t) &= U(t,0).
\end{aligned}
\tag{2.10}
$$

Note that $U(t',t)$ has a constant eigenvector ${}^{\mathrm{t}}[1,\dots,1]$ whose eigenvalue is unity,

$$
\begin{aligned}
&U(t',t)\,{}^{\mathrm{t}}[1,\dots,1] = {}^{\mathrm{t}}[1,\dots,1],\ \ [1,\dots,1]U(t',t) = [1,\dots,1] \\
&\Leftrightarrow U(t',t)T = TU(t',t) = T.
\end{aligned}
\tag{2.11}
$$

This property of $U(t',t)$ is a consequence of the sum-to-zero property of the matrix M in eq. (2.6).

Substitution of the formal solution (2.8) into eqs. (2.3) and (2.7) gives the following solution of the initial value problem of the Lax equation (2.2):

$$
\begin{aligned}
D(t) &= U^\dagger(0,t)\frac{\mathrm{e}^{\mathrm{i}\omega t}L^+(0) - \mathrm{e}^{-\mathrm{i}\omega t}L^-(0)}{2\mathrm{i}\omega}U(0,t), \\
L(t) &= U^\dagger(0,t)\frac{\mathrm{e}^{\mathrm{i}\omega t}L^+(0) + \mathrm{e}^{-\mathrm{i}\omega t}L^-(0)}{2}U(0,t),
\end{aligned}
\tag{2.12}
$$

which means that the eigenvalues of the time-dependent Hermitian matrix $\dfrac{\mathrm{e}^{\mathrm{i}\omega t}L^+(0) - \mathrm{e}^{-\mathrm{i}\omega t}L^-(0)}{2\mathrm{i}\omega}$ give the solution of the initial value problem of the Calogero model. The time-evolution unitary matrix $U(0,t)$, which has been formally introduced as the Dyson series of $M(t)$, is given here as the diagonalizing matrix.

The unitary matrix $U(0,t)$ provides a map of the Calogero model into the matrix-valued harmonic oscillator. Let us introduce the matrices

$$
\begin{aligned}
&X^\pm(t) := U(0,t)L^\pm(t)U^\dagger(0,t) = P(t) \pm \mathrm{i}\omega Q(t), \\
&P(t) := U(0,t)L(t)U^\dagger(0,t), \quad Q(t) := U(0,t)D(t)U^\dagger(0,t).
\end{aligned}
\tag{2.13}
$$

Substituting $X^\pm$ into the Lax equation (2.2), we have

$$
\frac{\mathrm{d}X^\pm}{\mathrm{d}t} = \pm\mathrm{i}\omega X^\pm,
\tag{2.14}
$$

which are equivalent to the equations of motion of the harmonic oscillator

$$
\frac{\mathrm{d}P}{\mathrm{d}t} = -\omega^2 Q, \quad \frac{\mathrm{d}Q}{\mathrm{d}t} = P.
\tag{2.15}
$$

Thus we confirm that the Lax equations of the Calogero model (2.2) can be mapped to the equations of motion of the matrix-valued harmonic oscillator, eqs. (2.14) and (2.15). Though the Hermitian matrices $P(t)$ and $Q(t)$ can possess $2N^2$ parameters for their initial values, their definitions (2.13) introduce the restriction in the initial values

$$
P(0) = L(0), \quad Q(0) = D(0),
\tag{2.16}
$$

whose number $2N$ is the same as that of the Calogero model. The solution of the initial value problem of the above equations of motion is given by

$$
\begin{aligned}
&Q(t) = D(0)\cos\omega t + \frac{L(0)}{\omega}\sin\omega t, \quad P(t) = L(0)\cos\omega t - \omega D(0)\sin\omega t. \\
&\left[\Leftrightarrow \ X^{\pm}(t) = \mathrm{e}^{\pm \mathrm{i}\omega t} L^{\pm}(0)\right]
\end{aligned}
\tag{2.17}
$$

Substitution of the above solution (2.17) into the definition of $P(t)$ and $Q(t)$ in eq. (2.13) reproduces the solution of the initial value problem of the Calogero model (2.12). In particular, the coordinates of the Calogero model is given by the eigenvalues of the matrix-valued harmonic oscillator $Q(t)$ in eq. (2.17). That is the essence of the projection method.

The restrictions on the initial values (2.16) can be explained in terms of the constraints on the variables $Q(t)$ and $P(t)$. From eqs. (2.6) and (2.13), one can derive

$$
\begin{aligned}
[Q(t), P(t)] &= U(0,t)[D(t), L(t)]U^{\dagger}(0,t) \\
&= \mathrm{i}aU(0,t)(T-E)U^{\dagger}(0,t).
\end{aligned}
$$

By use of the property of the time-evolution unitary matrix (2.11), one obtains

$$
[Q, P] = \mathrm{i}a(T-E), \tag{2.18}
$$

which poses $N(N-1)$ constraints on the "unconstrained" variables $Q(t)$ and $P(t)$ given by two Hermitian matrices whose off-diagonal elements are pure imaginaries, i.e.,

$$
\begin{aligned}
&Q_{ij}(t) := q_{ii}(t)\delta_{ij} + \mathrm{i}q_{ij}(t), \quad P_{ij}(t) := p_{ii}(t)\delta_{ij} + \mathrm{i}p_{ij}(t), \\
&q_{ii}(t),\ q_{ij}(t),\ p_{ii}(t),\ p_{ij}(t) \in \mathbb{R}, \quad q_{ij}(t) = -q_{ji}(t),\ p_{ij}(t) = -p_{ji}(t).
\end{aligned}
\tag{2.19}
$$

The above restriction (2.19) is consistent with the solution of the initial value problem (2.17). The number of the independent variables in $Q(t)$ and $P(t)$ given by eq. (2.19) is $N(N+1)$. One thus reproduces the number of the initial values $2N$ as the degrees of freedom of the constrained system, $N(N+1) - N(N-1) = 2N$. The transformation (2.13) thus should be interpreted as a map of the Calogero model into the harmonic oscillator (2.15) of the Hermitian matrix given by eq. (2.19) with the constraints (2.18).

III Integrable Discretization

As we have discussed in the previous section, the equations of motion (the Lax equation) of the Calogero model can be mapped to those of the matrix-valued harmonic oscillator. We thus begin with the integrable discretization of the matrix-valued harmonic oscillator, whose difference equations of motion are given as

$$
\begin{aligned}
\Delta_+ Q_n &= \frac{1}{2}(P_{n+1} + P_n), \\
\Delta_+ P_n &= -\frac{1}{2}\omega^2(Q_{n+1} + Q_n),
\end{aligned}
\tag{3.1}
$$

where Q_n and P_n are Hermitian matrices whose initial values are fixed as $Q_0 = D(0)$ and $P_0 = L(0)$ so as to relate them with the difference analogue of the Calogero model. The difference equations

given above have the same form as those for one-dimensional harmonic oscillator discretized by the energy conservation scheme [9], which is nothing but the implicit midpoint rule giving a symplectic integration method of order 2 (see Theorem VI.3.4 in ref. [1]).

As in eq. (2.13), we introduce the new variables

$$X_n^{\pm} := P_n \pm \mathrm{i}\omega Q_n.$$

This brings the difference equations of motion into

$$\Delta_+ X_n^{\pm} = \pm\frac{1}{2}\mathrm{i}\omega(X_{n+1}^{\pm} + X_n^{\pm}),$$

which is equivalent to

$$\Big(1 \mp \frac{1}{2}\mathrm{i}\omega\Delta t\Big)X_{n+1}^{\pm} = \Big(1 \pm \frac{1}{2}\mathrm{i}\omega\Delta t\Big)X_n^{\pm}. \tag{3.2}$$

Defining the rescaled time-step by

$$\Delta\tau := \frac{2}{\omega}\arctan\frac{\omega\Delta t}{2}, \tag{3.3}$$

the recursion relation (3.2) is rewritten as

$$X_{n+1}^{\pm} = \frac{1 \pm \frac{\mathrm{i}\omega\Delta t}{2}}{1 \mp \frac{\mathrm{i}\omega\Delta t}{2}}X_n^{\pm} = \mathrm{e}^{\pm\mathrm{i}\omega\Delta\tau}X_n^{\pm}.$$

Thus the solution of the initial value problem of the discrete harmonic oscillator is

$$X_n^{\pm} = \mathrm{e}^{\pm\mathrm{i}n\omega\Delta\tau}X_0^{\pm},$$

or

$$\begin{aligned} Q_n &= D(0)\cos n\omega\Delta\tau + \frac{1}{\omega}L(0)\sin n\omega\Delta\tau,\\ P_n &= L(0)\cos n\omega\Delta\tau - \omega D(0)\sin n\omega\Delta\tau, \end{aligned} \tag{3.4}$$

in terms of Q_n and P_n. The above solution (3.4) is exactly the same as that for the harmonic oscillator with the continuous time (2.17) up to time-rescale (3.3).

The recursion relation (3.2) provides us with an efficient way to construct the constants of motion of the discrete harmonic oscillator. Consider an arbitrary product of X_{n+1}^+ and X_{n+1}^- like $(X_{n+1}^+)^{l_1}(X_{n+1}^-)^{m_1}\cdots$, for $l_1, m_1 = 0, 1, 2\cdots$, and reverse the time for one discrete time-step using the recursion relation (3.2). Then one obtains

$$\begin{aligned} (X_{n+1}^+)^{l_1}(X_{n+1}^-)^{m_1}\cdots &= \mathrm{e}^{\mathrm{i}\omega l_1\Delta\tau}(X_n^+)^{l_1}\mathrm{e}^{-\mathrm{i}\omega m_1\Delta\tau}(X_n^-)^{m_1}\cdots\\ &= \mathrm{e}^{\mathrm{i}\omega(l_1+\cdots-(m_1+\cdots))\Delta\tau}(X_n^+)^{l_1}(X_n^-)^{m_1}\cdots. \end{aligned} \tag{3.5}$$

Taking the trace of the above relation, one obtains

$$\mathrm{Tr}\Big(\overrightarrow{\prod_j}(X_{n+1}^+)^{l_j}(X_{n+1}^-)^{m_j}\Big) = \exp\Big(\mathrm{i}\omega\Delta\tau\sum_k(l_k - m_k)\Big)\mathrm{Tr}\Big(\overrightarrow{\prod_j}(X_n^+)^{l_j}(X_n^-)^{m_j}\Big).$$

Thus one can construct constants of motion of the discrete harmonic oscillator in a way parallel to what is done for the Calogero model (2.4) by

$$\prod_i \mathrm{Tr}\Big(\overrightarrow{\prod_j}(X_n^+)^{l_{i,j}}(X_n^-)^{m_{i,j}}\Big), \qquad \text{when } \sum_{i,j} l_{i,j} = \sum_{i,j} m_{i,j}$$

as well as the matrix-valued constants of motion by

$$(X_{n+1}^+)^{l_1}(X_{n+1}^-)^{m_1}\cdots = (X_n^+)^{l_1}(X_n^-)^{m_1}\cdots,$$

when $l_1 + \cdots = m_1 + \cdots$. In particular, the following quantity,

$$[X_n^+, X_n^-] = -2\omega a(T-E)\ \big[= [X_0^+, X_0^-]\big] \tag{3.6}$$

is conserved since it is a special case of the above matrix-valued constants of motion. Note that the constants of motion of the matrix-valued harmonic oscillator with continuous time can be given by the same formulas.

As we have confirmed, the solution of the harmonic oscillator with discrete time (3.4) and that with the continuous time (2.17) agree up to time-rescale (3.3). Thus the eigenvalue of Q_n must trace the same trajectory of the Calogero model with continuous time. We shall discuss it more in detail.

Since the relations $Q_n = Q(n\Delta\tau)$ and $P_n = P(n\Delta\tau)$ hold as a consequence of eqs. (2.17), (3.3) and (3.4), we also have analogous relations with (2.13) for Q_n and P_n,

$$\begin{aligned} &D_n = U_n^\dagger Q_n U_n = D(n\Delta\tau),\ \ L_n = U_n^\dagger P_n U_n = L(n\Delta\tau),\\ &(D_n)_{ij} := x_{i,n}\delta_{ij},\ \ (L_n)_{ij} := p_{i,n}\delta_{ij} + \frac{\mathrm{i}a}{x_{i,n}-x_{j,n}}(1-\delta_{ij}), \end{aligned} \tag{3.7}$$

where the unitary matrix U_n is also given by the corresponding matrix in the theory for the model with continuous time:

$$U_n = U(0, n\Delta\tau). \tag{3.8}$$

Introduce $L_n^\pm$ in analogy with $L^\pm$ in eq. (2.3), i.e.,

$$L_n^\pm := L_n \pm \mathrm{i}\omega D_n = U_n^\dagger X_n^\pm U_n \tag{3.9}$$

and substitute it into the recursion relation (3.2). Then one obtains

$$\Big(1 \mp \frac{1}{2}\mathrm{i}\omega\Delta t\Big) U_{n+1} L_{n+1}^\pm U_{n+1}^\dagger = \Big(1 \pm \frac{1}{2}\mathrm{i}\omega\Delta t\Big) U_n L_n^\pm U_n^\dagger.$$

By multiplying with $U_n^\dagger$ on the left and U_{n+1} on the right, one obtains a recursion relation of the matrices $L_n^\pm$

$$\begin{aligned} &\Big(1 \mp \frac{1}{2}\mathrm{i}\omega\Delta t\Big) S_n L_{n+1}^\pm = \Big(1 \pm \frac{1}{2}\mathrm{i}\omega\Delta t\Big) L_n^\pm S_n,\\ &S_n := U_n^\dagger U_{n+1}, \end{aligned} \tag{3.10}$$

which we shall call the discrete Lax equations, or in short, the dLax equations. They will play an essential role in the construction of the constants of motion for the discrete time model. From the definition of the unitary matrix S_n and using the semigroup property of the time evolution unitary matrix $U(t', t)$ (2.10), one obtains

$$\begin{aligned} &S_n := U_n^\dagger U_{n+1} = U(n\Delta\tau, (n+1)\Delta\tau) = S_n[\boldsymbol{x}_n, \boldsymbol{p}_n; a, \omega; \Delta t], \\ &U_n = S_1 S_2 \cdots S_{n-1}, \quad U_0 = E. \end{aligned} \tag{3.11}$$

This indicates that S_n is the one-step time-evolution matrix. We should note that *the discrete inhomogeneous Lax's equation* introduced for the Calogero–Moser model (the Hamiltonian (1.1) with $\omega = 0$) in ref. [21] inspired us with the above recursion equation (3.10). However, the explicit forms of the Lax pair and the derivation of the recursion relation of this paper are different from those in ref. [21].

The recursion relation (3.10) can be interpreted as the discrete time analogue of the Lax equation of the Calogero model because of the following two reasons. The first reason is that the dLax equations (3.10) reduces to the Lax equations of the Calogero model (2.2) in the continuous time limit, $\Delta t \to 0$. The other reason is that the dLax equations also conserve the constants of motion of the Calogero model with continuous time. We shall discuss them more in detail.

From the Taylor expansions of $L_{n+1} = L((n+1)\Delta\tau)$ and $S_n = U(t' = n\Delta\tau, t = (n+1)\Delta\tau)$ together with the expression in the formal Dyson series (2.9) at $t = n\Delta\tau$, one obtains

$$\begin{aligned} &L_{n+1}^{\pm} \sim L_n^{\pm} + \frac{\mathrm{d}L_n^{\pm}}{\mathrm{d}t}\Delta t + O(\Delta t^2), \\ &S_n \sim E + M_n \Delta t + O(\Delta t^2), \quad M_n := M(n\Delta\tau). \end{aligned} \tag{3.12}$$

Substitution of the above expressions into the dLax equations (3.10) yields

$$(1 \mp \frac{1}{2}\mathrm{i}\omega\Delta t)(E + M_n\Delta t)(L_n^{\pm} + \frac{\mathrm{d}L_n^{\pm}}{\mathrm{d}t}\Delta t) = (1 \pm \frac{1}{2}\mathrm{i}\omega\Delta t)L_n^{\pm}(E + M_n\Delta t).$$

Dividing the above relation by Δt and taking the limit $\Delta t \to 0$, one gets

$$\frac{\mathrm{d}L_n^{\pm}}{\mathrm{d}t} = \left[L_n^{\pm}, M_n\right] \pm \mathrm{i}\omega L_n^{\pm},$$

which is nothing but the Lax equation of the Calogero model with continuous time.

In a way parallel to the construction of the constants of motion of the discrete harmonic oscillator, one can construct the constants of motion of the dLax equations (3.10). Using the definition (3.9) of $L_n^{\pm}$ in eq. (3.5) and multiplying by $U_n^\dagger$ and U_{n+1} respectively from the left and the right, one gets

$$\begin{aligned} S_n(L_{n+1}^{+})^{l_1}(L_{n+1}^{-})^{m_1}\cdots &= \mathrm{e}^{\mathrm{i}\omega l_1\Delta\tau}(L_n^{+})^{l_1}\mathrm{e}^{-\mathrm{i}\omega m_1\Delta\tau}(L_n^{-})^{m_1}\cdots S_n \\ &= \mathrm{e}^{\mathrm{i}\omega(l_1+\cdots-(m_1+\cdots))\Delta\tau}(L_n^{+})^{l_1}(L_n^{-})^{m_1}\cdots S_n. \end{aligned}$$

Thus one has

$$(L_{n+1}^{+})^{l_1}(L_{n+1}^{-})^{m_1}\cdots = \mathrm{e}^{\mathrm{i}\omega(l_1+\cdots-(m_1+\cdots))\Delta\tau}S_n^\dagger(L_n^{+})^{l_1}(L_n^{-})^{m_1}\cdots S_n.$$

Since the trace of an arbitrary product of matrices is invariant under any cyclic change of the order of the matrices, one has

$$\mathrm{Tr}\Big(\overrightarrow{\prod_j}(L_{n+1}^+)^{l_j}(L_{n+1}^-)^{m_j}\Big) = \exp\Big(\mathrm{i}\omega\Delta\tau\sum_k(l_k - m_k)\Big)\mathrm{Tr}\Big(\overrightarrow{\prod_j}(L_n^+)^{l_j}(L_n^-)^{m_j}\Big),$$

which leads to

$$\prod_i \mathrm{Tr}\Big(\overrightarrow{\prod_j}(L_{n+1}^+)^{l_{i,j}}(L_{n+1}^-)^{m_{i,j}}\Big) = \exp\Big(\mathrm{i}\omega\Delta\tau\sum_{k,l}(l_{k,l} - m_{k,l})\Big)\prod_i \mathrm{Tr}\Big(\overrightarrow{\prod_j}(L_n^+)^{l_{i,j}}(L_n^-)^{m_{i,j}}\Big).$$

One thus concludes that

$$\prod_i \mathrm{Tr}\Big(\overrightarrow{\prod_j}(L_n^+)^{l_{i,j}}(L_n^-)^{m_{i,j}}\Big) \tag{3.13}$$

gives the constant of motion of the dLax equations (3.10) when

$$\sum_{i,j} l_{i,j} = \sum_{i,j} m_{i,j}.$$

This is in complete agreement with the situation for the Calogero model with continuous time (2.4).

Since $L_n^\pm$ should have the same form as $L^\pm$ for the continuous time model, the commutator between L_n^+ and L_n^- also should be a constant matrix as in eq. (2.6). In the continuous time theory, this constant matrix is associated with the nontrivial constraints of the matrix-valued harmonic oscillator (2.18). One can observe the same situations in our discrete time model (3.10). Substitution of eq. (3.9) into $[L_n^+, L_n^-]$ with the help of eq. (3.6) yields

$$\begin{aligned}[L_n^+, L_n^-] &= U_n^\dagger[X_n^+, X_n^-]U_n \\ &= U_n^\dagger\big(-2\omega a(T-E)\big)U_n.\end{aligned}$$

Since U_n and also S_n are made from the time-evolution unitary matrix $U(t', t)$ as in eqs. (3.8) and (3.11), they also have a constant eigenvector ${}^{\mathrm{t}}[1, \ldots, 1]$ whose eigenvalues are unity,

$$\begin{aligned} &U_n{}^{\mathrm{t}}[1,\ldots,1] = {}^{\mathrm{t}}[1,\ldots,1],\ [1,\ldots,1]U_n = [1,\ldots,1] \Leftrightarrow U_nT = TU_n = T, \\ &S_n{}^{\mathrm{t}}[1,\ldots,1] = {}^{\mathrm{t}}[1,\ldots,1],\ [1,\ldots,1]S_n = [1,\ldots,1] \Leftrightarrow S_nT = TS_n = T. \end{aligned} \tag{3.14}$$

Thus one immediately obtains

$$[L_n^+, L_n^-] = -2\omega a(T - E). \tag{3.15}$$

using the property (3.14) of U_n. The relation (3.15) is also equivalent to

$$[D_n, L_n] = \mathrm{i}a(T - E).$$

Note that eq. (3.7) provides the most general form of L_n that satisfies the above relation together with the diagonal matrix $(D_n)_{ij} = x_i\delta_{ij}$.

IV Difference Equations of Motion

Though we have developed an integrable discretization for the Calogero model in the previous section, the difference equations of motion we have obtained is nevertheless a formal expression. An explicit expression for S_n is necessary in order to obtain the dLax equations for the Calogero model explicitly. To do this, we restrict the number of particles to two in the discussion of the explicit form of the S_n matrix.

From the dLax equations (3.10), one obtains

$$L_{n+1} \pm \mathrm{i}\omega D_{n+1} := L^{\pm}_{n+1} = \frac{1 \pm \frac{1}{2}\mathrm{i}\omega\Delta t}{1 \mp \frac{1}{2}\mathrm{i}\omega\Delta t} S_n^{\dagger} L_n^{\pm} S_n.$$

Solution of the above equations with respect to D_{n+1} and L_{n+1} are given by

$$\begin{aligned} D_{n+1} &= \frac{1}{2\mathrm{i}\omega} S_n^{\dagger}\left[\frac{1 + \frac{1}{2}\mathrm{i}\omega\Delta t}{1 - \frac{1}{2}\mathrm{i}\omega\Delta t} L_n^{+} - \frac{1 - \frac{1}{2}\mathrm{i}\omega\Delta t}{1 + \frac{1}{2}\mathrm{i}\omega\Delta t} L_n^{-}\right] S_n, \\ L_{n+1} &= \frac{1}{2} S_n^{\dagger}\left[\frac{1 + \frac{1}{2}\mathrm{i}\omega\Delta t}{1 - \frac{1}{2}\mathrm{i}\omega\Delta t} L_n^{+} + \frac{1 - \frac{1}{2}\mathrm{i}\omega\Delta t}{1 + \frac{1}{2}\mathrm{i}\omega\Delta t} L_n^{-}\right] S_n. \end{aligned} \tag{4.1}$$

Since D_{n+1} is diagonal, S_n diagonalizes

$$\frac{1 + \frac{1}{2}\mathrm{i}\omega\Delta t}{1 - \frac{1}{2}\mathrm{i}\omega\Delta t} L_n^{+} - \frac{1 - \frac{1}{2}\mathrm{i}\omega\Delta t}{1 + \frac{1}{2}\mathrm{i}\omega\Delta t} L_n^{-}.$$

S_n thus can be constructed from the eigenvectors $\boldsymbol{v}_{i,n}$, $i = 1, \cdots, N$ of the above matrix. Normalized eigenvectors $\boldsymbol{s}_{i,n}$ that constitute $S_n = [\boldsymbol{s}_{1,n} \cdots \boldsymbol{s}_{N,n}]$ should have the following form,

$$\boldsymbol{s}_{i,n} = \frac{\boldsymbol{v}_{i,n}}{\sum_{j=1}^{N} (\boldsymbol{v}_{i,n})_j} \Rightarrow \sum_{j=1}^{N} (\boldsymbol{s}_{i,n})_j = 1 \tag{4.2}$$

so as to satisfy the condition (3.14). Ordering of the eigenvectors $\boldsymbol{s}_{i,n}$ is uniquely determined by the Taylor expansion of S_n (3.12), or equivalently

$$\lim_{\Delta t \to 0} S_n = E. \tag{4.3}$$

Considering the eqs. (4.1) (4.2) and (4.3) all together, one confirms that

$$\begin{aligned} S_n &= E + \frac{2a\mathrm{i}\Delta t}{Y_n - 2a\mathrm{i}\Delta t + \sqrt{4a^2\Delta t^2 + Y_n^2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= E + (\mathrm{i}M_{\mathrm{i},n}\Delta t - M_{\mathrm{r},n}\Delta t^2) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim E + M_n \Delta t \quad (\Delta t \to 0) \end{aligned} \tag{4.4}$$

gives the explicit form of the S_n matrix of the dLax equations for the Calogero model for the two-body case. It is straightforward to verify that the dLax equations (3.10) with the above S_n matrix (4.4) yields the difference equations of motion (1.2) shown in the introduction.

In order to confirm how well the dLax equations (3.10) with the S_n matrix (4.4) or the difference equations of motion (1.2) of the Calogero model with discrete time trace the behavior of that with

continuous time, the explicit expression of the analytic solution for the continuous-time model is of great help, which is given by the eigenvalues of $Q(t)$ in eq. (2.17). But there is another derivation. Since $U(t',t)$ is related to S_n according to eq. (3.11) with $t' := n\Delta\tau$ and $t := (n+1)\Delta\tau$, one can obtain the explicit form of $U(t',t)$ for the two-body case from that of S_n (4.4):

$$
\begin{aligned}
U(t',t) &= E + \frac{\frac{2ai\sin\omega(t-t')}{\omega}}{\mathcal{Y} - \frac{2ai\sin\omega(t-t')}{\omega} + \sqrt{\frac{4a^2\sin^2\omega(t-t')}{\omega^2} + \mathcal{Y}^2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\
\mathcal{Y} &:= (x_1 - x_2)^2 \cos\omega(t-t') + (p_1 - p_2)(x_1 - x_2)\frac{\sin\omega(t-t')}{\omega}.
\end{aligned} \tag{4.5}
$$

Substitution of the explicit forms of $U(0,t)$ (4.5) and $L^{\pm}(0)$ for the two-body case into eq. (2.12), one obtains the explicit form of the analytic solution of the two-body Calogero model with continuous time:

$$
\begin{aligned}
x_1(t) =& \frac{1}{2}\Big[\big(x_{1,0} + x_{2,0}\big)\cos\omega t + \frac{1}{\omega}\big(p_{1,0} + p_{2,0}\big)\sin\omega t\Big] \\
&+ \frac{1}{2(x_{1,0} - x_{2,0})}\sqrt{\frac{4a^2\sin^2\omega t}{\omega^2} + \Big[(x_{1,0} - x_{2,0})^2\cos\omega t + (p_{1,0} - p_{2,0})(x_{1,0} - x_{2,0})\frac{\sin\omega t}{\omega}\Big]^2} \\
=&: x_1[\boldsymbol{x}_0, \boldsymbol{p}_0; a, \omega; t] \\
p_1(t) =& \frac{1}{2}\Big[\big(p_{1,0} + p_{2,0}\big)\cos\omega t - \omega\big(x_{1,0} + x_{2,0}\big)\sin\omega t\Big] \\
&+ \frac{1}{2(x_{1,0} - x_{2,0})\sqrt{\frac{4a^2\sin^2\omega t}{\omega^2} + \Big[(x_{1,0} - x_{2,0})^2\cos\omega t + (p_{1,0} - p_{2,0})(x_{1,0} - x_{2,0})\frac{\sin\omega t}{\omega}\Big]^2}} \\
&\times \Bigg[\frac{2a^2\sin 2\omega t}{\omega} + \Big[(x_{1,0} - x_{2,0})^2\cos\omega t + (p_{1,0} - p_{2,0})(x_{1,0} - x_{2,0})\frac{\sin\omega t}{\omega}\Big] \\
&\times \Big[-\omega(x_{1,0} - x_{2,0})^2\sin\omega t + (p_{1,0} - p_{2,0})(x_{1,0} - x_{2,0})\cos\omega t\Big]\Bigg] \\
=&: p_1[\boldsymbol{x}_0, \boldsymbol{p}_0; a, \omega; t] \\
x_2(t) =& x_1(t)\Big|_{1\leftrightarrow 2}, \quad p_2(t) = p_1(t)\Big|_{1\leftrightarrow 2}
\end{aligned} \tag{4.6}
$$

The above expressions are used to numerically display the exact analytic results in the following figures. The initial values, the interaction parameter, the strength of the external harmonic well are set at $(x_1, p_1, x_2, p_2)|_{t=0} = (-4.00, 5.00, 2.00, 1.00)$, $a = 3.00$ and $\omega = 0.314$ throughout the numerical calculation in the following.

Figure 1 presents the time evolution of the coordinates and the momenta generated by the analytic solution and the difference equations of motion (1.2) that gives the super-integrable discretization. The time-step in the super-integrable discretization is set at $\Delta t = 1.00$. The relation between the time t and the number of iterations n is given by $n = t/\Delta\tau$ where $\Delta\tau = 0.991903$. The time interval $240 \times \frac{2\pi}{\omega} = 4802 \le t \le 4842 = 242 \times \frac{2\pi}{\omega}$ corresponds to $4841 \le n \le 4881$ in terms of the number of iterations for the discrete case. As has been confirmed by the correspondence between the matrices L and D for the super-integrable discretization and those for the continuous time model (3.7), the solution of the discrete equations of motion (1.2) gives a

sequence of canonical variables that are "sampled" from the orbit of the exact analytic solution, i.e.,

$$x_{i,n} = x_i(t = n\Delta\tau), \;\; p_{i,n} = p_i(t = n\Delta\tau). \tag{4.7}$$

Even though the coordinates and the momenta generated by the difference equations of mo-

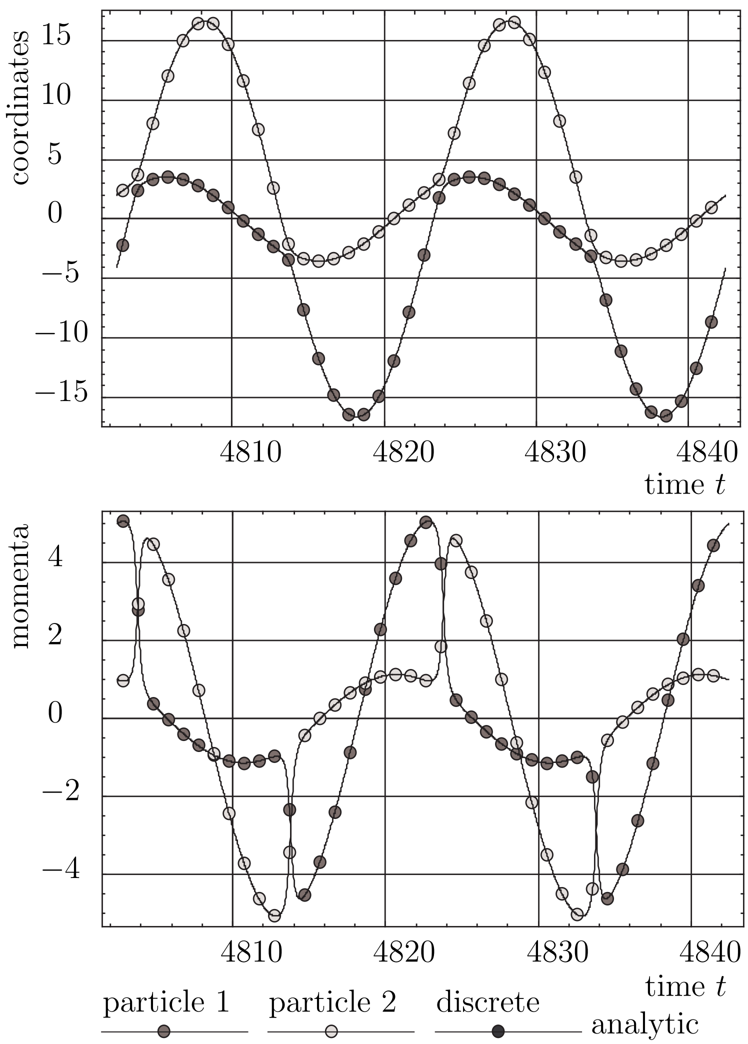

Figure 1: The time evolution of the coordinates and the momenta generated by the analytic solution and the difference equations of motion (1.2).

tion (1.2) are computed through sufficiently large numbers of iterations, they are in quite good agreement with those of the exact analytic solution, which gives a good numerical confirmation of the correspondence (4.7).

One can confirm the precise agreement of the two solutions in a more reliable manner by observing the relative errors of the constants of motion. Using eq. (2.4) (or eq. (3.13)), we have already given three constants of motion of the Calogero model (2.5). Note that C_1 and I_1 correspond to the Hamiltonian for the center of mass and the Calogero Hamiltonian, respectively.

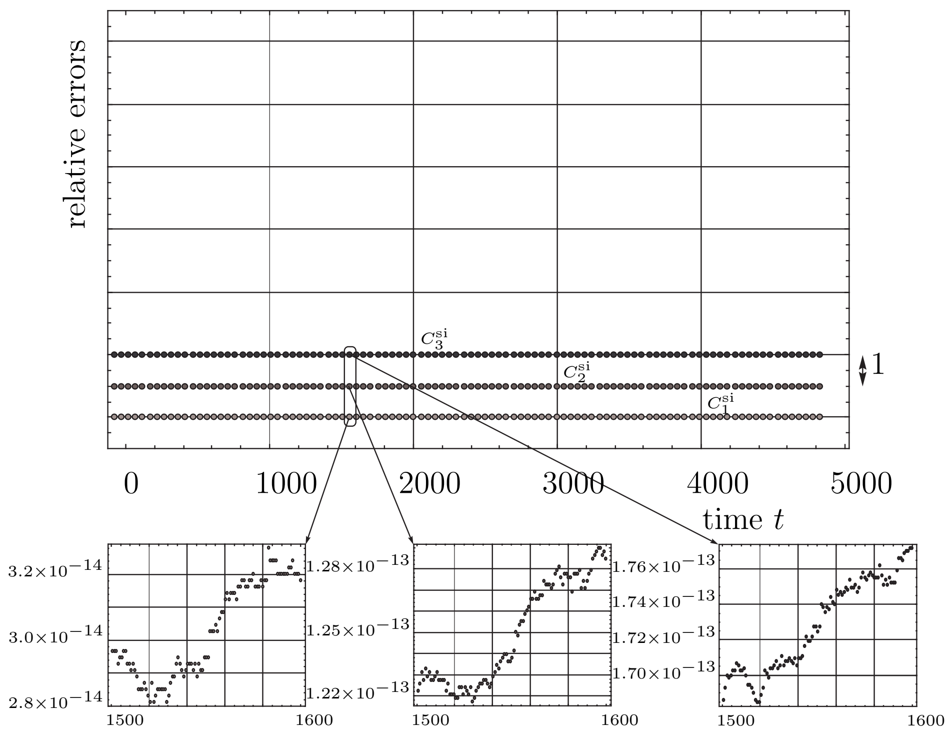

Figure 2: The relative errors of the constants of motion $C_i^{\rm si} := C_{i,n}/C_{i,0}$, $n = t/\Delta\tau$, of the discrete solutions given by the super-integrable discretization (1.2).

Those three constants of motion give a set of independent conserved quantities. But just for convenience, we introduce a set of slightly modified constants of motion:

$$\begin{aligned} C_1 &= (p_1+p_2)^2 + \omega^2(x_1+x_2)^2, \\ C_2 &:= 2I_1 - C_1 = (p_1-p_2)^2 + \omega^2(x_1-x_2)^2 + \frac{4a^2}{(x_1-x_2)^2}, \\ C_3 &:= \frac{C_1^2 - I_2}{4\omega^2} = (x_1p_2 - x_2p_1)^2 + \frac{2a^2(x_1^2+x_2^2)}{(x_1-x_2)^2}. \end{aligned} \tag{4.8}$$

Note that C_2 is the Hamiltonian for the relative coordinates and C_3 can be interpreted as a "modified quadratic angular momentum." We use the above C_i's, $i = 1, 2, 3$, for numerical calculation.

Figure 2 presents the relative errors of the constants of motion $C_i^{\rm si} := C_{i,n}/C_{i,0}$, $n = t/\Delta\tau$ of the discrete solutions given by the super-integrable discretization (1.2). The offsets ($= (i-1)\times 1.0$) are added for convenience of presentation. Even though one sees the growth of errors caused by round-off errors that is inevitable in any numerical analysis, one confirms that all the constants of motion are conserved with a precision of the machine epsilon times the number of iterations. Thus one concludes that fig. 2 presents a direct numerical verification of the fact that the difference equation of motion (1.2) preserves the super-integrability of the Calogero model.

Let us compare the above results with numerical demonstrations by two other discretizations.

The first one is the energy conservation scheme (see pp. 159–161 in ref. [1]),

$$\begin{aligned}
\Delta_+ x_{i,n} &= \frac{1}{2}(p_{i,n+1} + p_{i,n}),\\
\Delta_+ p_{i,n} &= -\frac{1}{2}\omega^2(x_{i,n+1} + x_{i,n}) + \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{a^2\big((x_{i,n+1} - x_{j,n+1}) + (x_{i,n} - x_{j,n})\big)}{(x_{i,n+1} - x_{j,n+1})^2(x_{i,n} - x_{j,n})^2},
\end{aligned} \tag{4.9}$$

which keeps two constants of motion C_1 and I_1 (consequently, C_2) exactly for arbitrary number of particles N. As one can see, the scheme (4.9) is an implicit scheme, which involves numerical solution of simultaneous algebraic equations. The other scheme is the symplectic Euler method (see Theorem VI.3.3 in ref. [1]),

$$\begin{aligned}
\Delta_+ x_{i,n} &= p_{i,n},\\
\Delta_+ p_{i,n} &= -\omega^2 x_{i,n+1} + \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{2a^2}{(x_{i,n+1} - x_{j,n+1})^3},
\end{aligned} \tag{4.10}$$

which is an explicit scheme. Note that Δ_+ denotes the advanced time-difference (1.3).

Figures 3 and 4 present the relative errors $C_i^{\mathrm{ec,se}} := C_{i,n}/C_{i,0}$, $n = t/\Delta t$ of the discrete solution given by the energy conservation scheme (ec) and the symplectic Euler method (se) for the two body case ($N = 2$). We should note that we do not present all the data of each relative errors to keep the size of the data file reasonable and that the offsets ($= (i-1) \times 1.0$) are added for convenience of presentation. The initial condition, the coupling parameter and the strength of the harmonic confinement are the same as those given for the numerical calculation that gives figs. 1 and 2. The discrete time step is set at $\Delta t = 0.200$ for these schemes.

One observes that the relative errors of the energies (C_1^{ec} and C_2^{ec}) calculated by the energy conservation scheme (4.9) remain within the order of the machine epsilon ($= 10^{-16}$) times the number of iteration ($= t/\Delta t$), which gives a numerical confirmation that the energies are conserved by the scheme. On the other hand, one also confirms that the relative errors (except for C_1^{ec} and C_2^{se}) are much larger than the order of machine epsilon times the number of iteration, even though one cannot observe growth of the relative errors in both two figures. These errors are not brought about by the round-off errors at the order of machine epsilon times the number of iteration, but by the schemes themselves. One perceives that there is a clear distinction between fig. 2 or the super-integrable discretization (1.2) and figs. 3 and 4 or the energy conservation scheme (4.9) and the symplectic Euler method (4.10).

One of the characteristics of the super-integrable system is that its bounded orbit is always closed in the phase space and, in particular, in the configuration space. As presented in fig. 5, the orbit in the x_1-x_2 plane, or the "Lissajous plot" in other words, thus provides the distinctest way of comparing the three discretizations. The time interval of the Lissajous plots is $0 \le t \le 200$, which corresponds to ten periods of motion. While the orbits generated by the energy conservation scheme and the symplectic Euler method are not closed, the orbit by the super-integrable discretization is always on that of the exact analytic solution, which is, of course, closed.

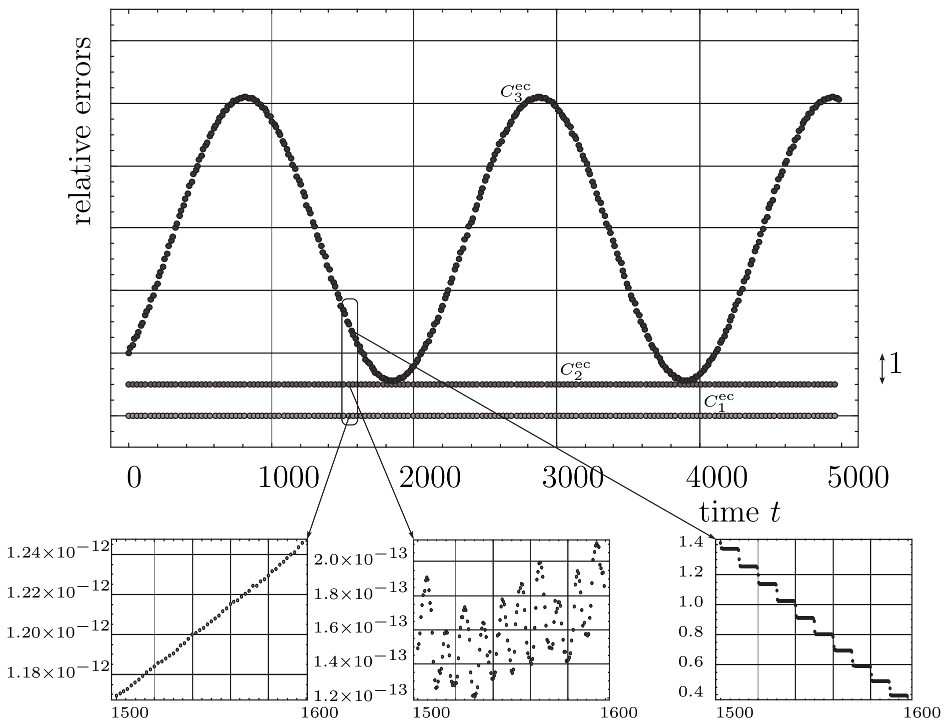

Figure 3: The relative errors of the constants of motion, $C_i^{\rm ec} := C_{i,n}/C_{i,0}$, $n = t/\Delta t$, of the discrete solution given by the energy conservation scheme (4.9).

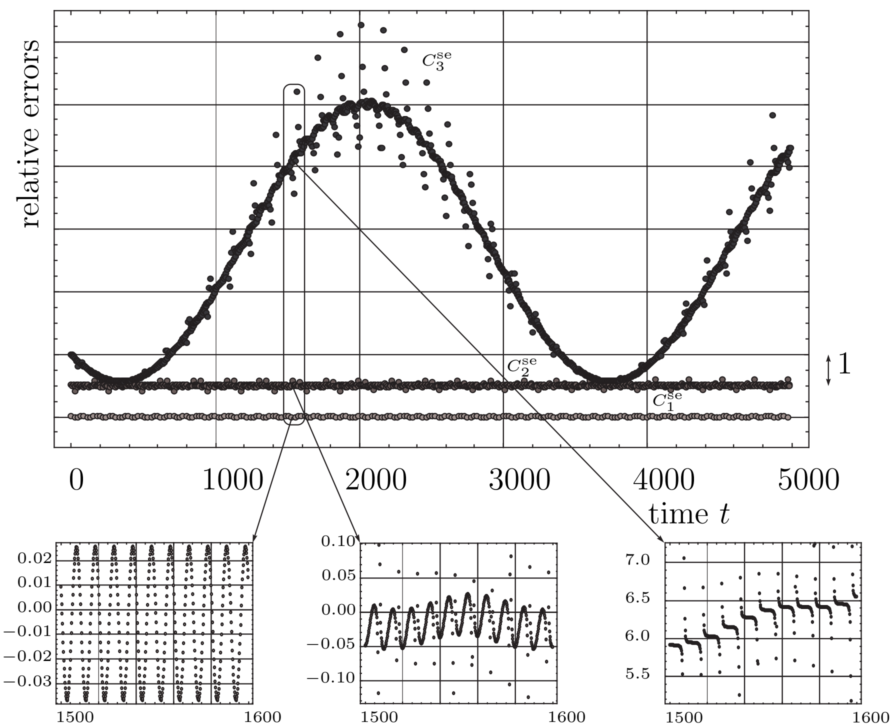

Figure 4: The relative errors of the constants of motion, $C_i^{\rm se} := C_{i,n}/C_{i,0}$, $n = t/\Delta t$, of the discrete solution given by the symplectic Euler method (4.10).

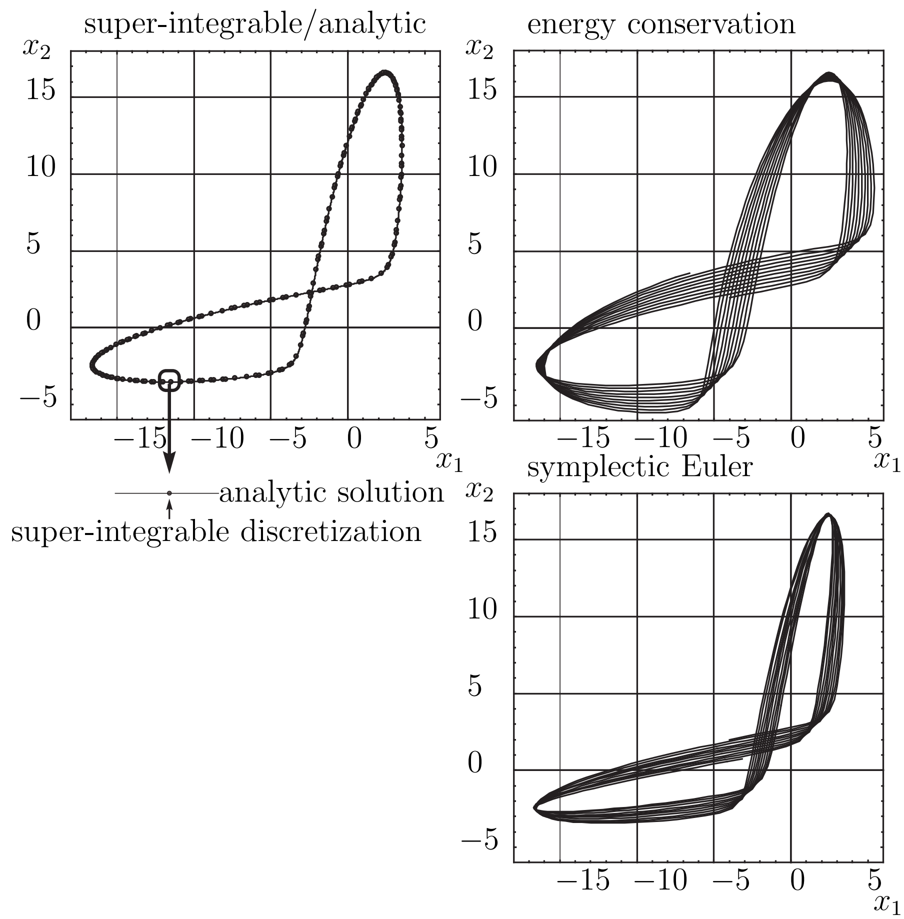

Figure 5: The “Lissajous plots”

V Summary and Concluding Remarks

The aim of the paper was to present a time-discretization for the Calogero model that maintains its super-integrable structure. As discussed in section II, the super-integrable structure of the original Calogero model is provided by the Lax formulation and the projection method, which are closely related to the equations of motion of the harmonic oscillator. With the help of the integrable discretization of the harmonic oscillator [9], the Lax formulation and the projection method are discretized. As a consequence, a time discretization that preserves the super-integrability of the Calogero model is presented in section III. In particular for the two-body case, an explicit form of the difference equations of motion (1.2) is obtained in section IV. The difference equations give an explicit scheme for time-integration. Numerical results by the super-integrable discretization (1.2) together with comparison with those by the energy conservation scheme (4.9) as well as the symplectic Euler method (4.10) are presented in five figures in section IV, which give a intuitive numerical verification of the super-integrability of the scheme.

Lastly, we should give several remarks on previous studies that are relevant to the present work. In refs. [21, 22, 23] (and also in ref. [6]), integrable discretizations of the rational Calogero–Moser model (the case $\omega = 0$) and its trigonometric, elliptic and "relativistic" (a q-difference generalization with respect to space coordinates) generalizations were presented. The structure of the projection method also underlies these integrable discretizations, but the interaction parameter of the continuous-time models and the time-step of the discrete-time models are related with each other in the integrable discretizations above. On the other hand, our discretization preserves not only the integrablity but also the super-integrablity of the Calogero model and both the time-step Δt and the interaction parameter a independently appear in the discrete equations of motion. Thus our discretization is apparently different from that of the previous studies. The possibility of the mutual penetration of the particles in the time-discrete model was reported in ref. [21], which is certainly unlike the continuous-time case. But the scheme given there is implicit and it could not privide a way to verify this possiblity. In our super-integrable discretization of the present work, however, we gave an explicit scheme of the Calogero model for the two-body case (1.2) and numerically observed in fig. 1 that there was no penetration of the particles. And this observation should be the same for the Calogero–Moser case corresponding to the limit $\omega \to 0$ of the present work. The comparison of the two different discretizations together with additional consideration on the super-integrability of the rational Calogero–Moser model [24] will be presented in a separate paper. Further studies on the trigonometric, elliptic and relativistic Calogero–Moser model along the line of our super-integrable discretization of the Calogero model as well as to obtain explicit forms of the difference equations of motion for general N-body case (or at least 3-body case) is worthy of interest.

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